JOURNAL OF APPROXMATION THEORY 75, 346–350 (1993)

Regularity of (0, 2) Interpolation*

YING GUANG SHI

Computing Center, Chinese Academy of Sciences, P.O. Box 2719, Beijig 100080, People's Republic of China

Communicated by Vilmos Totik

Received February 4, 1992; accepted in revised form August 3, 1992

In this paper some characterizations of the regularity of (0, 2) interpolation are given.

1. INTRODUCTION AND RESULT

Let us consider a triangular matrix A of nodes

$$1 \geqslant x_{1n} > x_{2n} > \cdots > x_{nn} \geqslant -1 \tag{1}$$

for $n = 2, 3, \ldots$. Let $\mathscr{P}_n$ be the set of polynomials of degree at most n. The problem of (0, 2) interpolation, which was initiated by Surányi and Turán [2], is, given a set of numbers y_{kn} and y''_{kn} to determine a polynomial $D_n(x; A) \in \mathscr{P}_{2n-1}$ (if any) satisfying

$$\begin{aligned} D_n(x_{kn}; A) &= y_{kn}, \qquad k = 1, 2, \ldots, n \\ D''_n(x_{kn}; A) &= y''_{kn}, \qquad k = 1, 2, \ldots, n. \end{aligned} \tag{2}$$

If for each n this problem has a unique solution for arbitrary sets of numbers y_{kn} and y''_{kn}, then the set of nodes (1) is said to be regular and $D_n(x; A)$ can be uniquely written as

$$D_n(x; A) = \sum_{k=1}^{n} [y_{kn} r_{kn}(x; A) + y''_{kn} \rho_{kn}(x; A)], \tag{3}$$

where $r_{kn}, \rho_{kn} \in \mathscr{P}_{2n-1}$ satisfy

$$r_{kn}(x_{jn}) = \delta_{kj}, \qquad r''_{kn}(x_{jn}) = 0, \qquad k, j = 1, 2, \ldots, n \tag{4}$$

* Supported by the National Natural Science Foundation of China.

0021-9045/93 $5.00

and

$$\rho_{kn}(x_{jn})=0, \qquad \rho''_{kn}(x_{jn})=\delta_{kj}, \qquad k, j=1, 2, ..., n. \tag{5}$$

One may ask: Do there exist n points with the property (1) for which all fundamental functions $\rho_k(x)$ of the second kind exist, but not all fundamental functions $r_k(x)$ of the first kind? Hwang [1] answers this question in an affirmative sense as follows.

THEOREM A. *For $n=4$, there are four points $\{x_j\}$, $j=1, 2, 3, 4$, satisfying (1) such that all $\rho_j(x)$ exist, but not all $r_j(x)$.*

We find that this result is not true. Indeed, we see from the last paragraph of his paper that in order to complete the proof of the theorem he should choose the constant c. But he did not do it. Moreover, he could not do it, because, conversely, we can give an answer to this question in a negative sense for every $n \geqslant 2$. That answer is the following

THEOREM. *Let $n \geqslant 2$ be fixed. Then the following statements are equivalent:*

(a) *The set of nodes (1) is regular.*

(b) *The matrix*

$$\mathscr{B}_n := [B''_k(x_j)]^n_{k,j=1} \tag{6}$$

is nonsingular, where

$$B_k(x)=(x-x_k)\, l^2_k(x), \qquad k=1, 2, ..., n \tag{7}$$

and l_k's are the fundamental functions of Lagrange interpolation.

(c) *There is one index k, $1 \leqslant k \leqslant n$, such that $r_k(x) \in \mathscr{P}_{2n-1}$ with the properties (4) exists uniquely.*

(d) *There is one index k, $1 \leqslant k \leqslant n$, such that $\rho_k(x) \in \mathscr{P}_{2n-1}$ with the properties (5) exists uniquely.*

(e) *There exists a set of polynomials $r_k(x) \in \mathscr{P}_{2n-1}$, $k=1, 2, ..., n$, with the properties (4).*

(f) *There exists a set of polynomials $\rho_k(x) \in \mathscr{P}_{2n-1}$, $k=1, 2, ..., n$, with the properties (5).*

Remark. Turán in [3] distinguishes "very good" matrices from "good" matrices: A matrix A is called "very good" if for every $n \geqslant 2$, the set (1) is regular; and a matrix A is "good" if for every $n \geqslant 2$, there exists at least one set of polynomials $r_k, \rho_k \in \mathscr{P}_{2n-1}$, $k=1, 2, ..., n$, with the properties (4) and (5), respectively. Now, according to our theorem, a "good" matrix must be "very good."

2. Proof of Theorem

The proof is accomplished by the implications (a) $\Rightarrow$ (c) $\Rightarrow$ (b) $\Rightarrow$ (a), (a) $\Rightarrow$ (d) $\Rightarrow$ (b) $\Rightarrow$ (a), (a) $\Leftrightarrow$ (e), and (a) $\Leftrightarrow$ (f). It is easy to see that (a) $\Rightarrow$ (c), (d), (e), and (f), respectively.

First we introduce the fundamental functions A_k, $B_k \in \mathscr{P}_{2n-1}$ of the first and the second kind for (0, 1) interpolation (they always exist):

$$A_k(x_j)=\delta_{kj}, \qquad A_k'(x_j)=0, \qquad k, j=1, 2, ..., n \tag{8}$$

and

$$B_k(x_j)=0, \qquad B_k'(x_j)=\delta_{kj}, \qquad k, j=1, 2, ..., n. \tag{9}$$

We have the following well-known facts:

(I) The B_k's are of the form (7).

(II) If $P \in \mathscr{P}_{2n-1}$, then P can be uniquely written as

$$P(x)=\sum_{i=1}^{n} [P(x_i) A_i(x)+P'(x_i) B_i(x)]. \tag{10}$$

(c) $\Rightarrow$ (b) If r_k exists uniquely for some k, then r_k can be uniquely written as

$$r_k(x)=A_k(x)+\sum_{i=1}^{n} r_k'(x_i) B_i(x). \tag{11}$$

Since $r_k''(x_j)=0$ for $j=1, 2, ..., n$,

$$\sum_{i=1}^{n} r_k'(x_i) B_i''(x_j)=-A_k''(x_j), \qquad j=1, 2, ..., n. \tag{12}$$

This system of equations has a unique solution if and only if the coefficient matrix $\mathscr{B}_n$ is nonsingular, as stated.

(d) $\Rightarrow$ (b) If ρ_k exists uniquely for some k, then ρ_k can be uniquely written as

$$\rho_k(x)=\sum_{i=1}^{n} \rho_k'(x_i) B_i(x) \tag{13}$$

and the system of equations

$$\sum_{i=1}^{n} \rho_k'(x_i)\, B_i''(x_j) = \delta_{kj}, \qquad j = 1, 2, ..., n \tag{14}$$

has a unique solution, which is equivalent to the nonsingularity of $\mathscr{B}_n$.

(b)$\Rightarrow$(a) If the matrix $\mathscr{B}_n$ is nonsingular, then the system of equations

$$\sum_{i=1}^{n} a_{ik} B_i''(x_j) = -A_k''(x_j), \qquad j = 1, 2, ..., n \tag{15}$$

has a unique solution $a_{1k}, a_{2k}, ..., a_{nk}$. Thus

$$r_k(x) = A_k(x) + \sum_{i=1}^{n} a_{ik} B_i(x) \tag{16}$$

is a unique polynomial in $\mathscr{P}_{2n-1}$ satisfying (4).

Similarly, if the matrix $\mathscr{B}_n$ is nonsingular, then the system of equations

$$\sum_{i=1}^{n} b_{ik} B_i''(x_j) = \delta_{kj}, \qquad j = 1, 2, ..., n \tag{17}$$

has a unique solution $b_{1k}, b_{2k}, ..., b_{nk}$. Thus

$$\rho_k(x) = \sum_{i=1}^{n} b_{ik} B_i(x) \tag{18}$$

is a unique polynomial in $\mathscr{P}_{2n-1}$ satisfying (5).

Then for arbitrary sets of numbers y_k and y_k'' there exists a unique polynomial $D_n(x; A) \in \mathscr{P}_{2n-1}$ with the form (3), which satisfies (2).

(e)$\Rightarrow$(a) Assume that $r_k(x)$ satisfies (4) for $k = 1, 2, ..., n$. Then both (11) and (12) hold for $k = 1, 2, ..., n$.

On the other hand, putting $P = x^m$ for $m = 2, 3, ..., n+1$ in (10) we have

$$x^m = \sum_{i=1}^{n} [x_i^m A_i(x) + m x_i^{m-1} B_i(x)], \qquad m = 2, 3, ..., n+1. \tag{19}$$

Differentiating twice and putting $x = x_j$ for $j = 1, 2, ..., n$ by (12) yield

$$\begin{aligned} m(m-1)\, x_j^{m-2} &= \sum_{i=1}^{n} [x_i^m A_i''(x_j) + m x_i^{m-1} B_i''(x_j)] \\ &= \sum_{k=1}^{n} x_k^m A_k''(x_j) + m \sum_{i=1}^{n} x_i^{m-1} B_i''(x_j) \\ &= \sum_{k=1}^{n} x_k^m \left[-\sum_{i=1}^{n} r_k'(x_i)\, B_i''(x_j) \right] + m \sum_{i=1}^{n} x_i^{m-1} B_i''(x_j) \\ &= \sum_{i=1}^{n} \left[m x_i^{m-1} - \sum_{k=1}^{n} x_k^m r_k'(x_i) \right] B_i''(x_j). \end{aligned}$$

Thus

$$x_j^{m-2} = \frac{1}{m(m-1)} \sum_{i=1}^{n} \left[m x_i^{m-1} - \sum_{k=1}^{n} x_k^m r_k'(x_i) \right] B_i''(x_j),$$

$$j = 1, 2, \ldots n, \quad m = 2, 3, \ldots, n+1.$$

This shows that for every $m = 2, 3, \ldots, n+1$ the vector $[x_1^{m-2}, x_2^{m-2}, \ldots, x_n^{m-2}]$ of n space $\mathscr{R}_n$ belongs to the linear span of the set

$$\{[B_i''(x_1), B_i''(x_2), \ldots, B_i''(x_n)] : i = 1, 2, \ldots, n\}.$$

But, as we know, the n vectors $[x_1^{m-2}, x_2^{m-2}, \ldots, x_n^{m-2}]$, $m = 2, 3, \ldots, n+1$, are linear independent, since the determinant of the matrix consisting of these vectors, known as Vandermonde's determinant, is nonzero. Therefore the matrix $\mathscr{B}_n$ must be nonsingular.

(f) $\Rightarrow$ (a) Assume that $\rho_k(x)$ satisfies (5) for $k = 1, 2, \ldots, n$. Then both (13) and (14) hold for $k = 1, 2, \ldots, n$. This shows that for every $k = 1, 2, \ldots, n$ the vector $[\delta_{k1}, \delta_{k2}, \ldots, \delta_{kn}]$ of n space $\mathscr{R}_n$ belongs to the linear span of the set

$$\{[B_i''(x_1), B_i''(x_2), \ldots, B_i''(x_n)] : i = 1, 2, \ldots, n\},$$

which implies that the matrix $\mathscr{B}_n$ is nonsingular.

This completes the proof.

Acknowledgments

The author thanks the referees for many helpful suggestions and comments on improving the original manuscript.

References

1. J. S. Hwang, On a problem of Turán about the optimal (0, 2)-interpolation, *Acta Math. Acad. Sci. Hungar.* **40**, Nos. 3, 4 (1982), 191–196.
2. J. Surányi and P. Turán, Notes on interpolation, I, *Acta Math. Acad. Sci. Hungar.* **6** (1955), 67–80.
3. P. Turán, On some open problems of approximation theory, *J. Approx. Theory* **29** (1980), 23–85.